

Gravitational Self-Energy and Black Holes in Newtonian Physics

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Abstract: A definition of a Newtonian black hole is possible which incorporates the mass-energy equivalence from special relativity. However, exploiting a spherical double shell model, it will be shown that the ensuing gravitational self-energy and mass renormalization prevent the formation of such an object.

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1 Introduction

The possible existence of a celestial object so massive to hold back even light with its gravity goes back to the end of '700 [1, 2]. According to Newtonian physics a spherically symmetric distribution of a mass M inside a region of radius R centered at the origin, yields the gravitational potential (for $r \geq R$)

$$\Phi(r) = -G \frac{M}{r} \quad (1)$$

Hence the energy of a test-particle δm_0 settled on its surface is

$$\delta U(R) = -G \frac{M \delta m_0}{R} \quad (2)$$

If rays of light were constituted by a flux of tiny particles with a given kinetic energy (as was believed at the time) one would immediately get, from the conservation of the mechanical energy (kinetic+potential), the condition for the mass M to be heavy enough to prevent light to escape from its surface. This condition defines a “Newtonian black hole” (NBH).

It is a widespread opinion that an “up-to-date” definition of a NBH is possible if one plugs Einstein’s special relativity into Newtonian gravitation. Indeed, taking into account the mass-energy equivalence together with the inertial-gravitational mass equality, one may write for the total mass $M_t(R)$ of the system (heavy mass M + test-particle δm_0 on its surface)

$$M_t(R) = M + \delta m_0 + \delta U(R)/c^2 = M + \delta m_0(1 + \Phi(R)/c^2) \quad (3)$$

For a relativistic particle (3) is supposed to hold as well, provided $\delta m_0 c^2$ represents the full relativistic energy of the particle. For a photon it is: $\delta m_0 = \hbar \omega / c^2$.

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Now if

$$\delta U(R) = -\delta m_0 c^2 \quad (4)$$

one has from (3) $M_t(R) = M$, i.e. the total energy of the system with or without δm_0 is the same. This means, for example, that a photon, leaving the surface of that sphere, must spend its whole energy $\hbar\omega$ to get out from the gravitational field and will end its journey with a vanishingly small frequency irrespective of the initial one. Therefore (4) is the up-to-date condition for the existence of a NBH. A given mass M_0 confined in a sphere of sufficiently small radius R_0 :

$$R_0 = GM_0/c^2 \quad (5)$$

leads to (4). If $R < R_0$ the meaning of R_0 is the maximum radial distance from where light cannot escape and corresponds to the so called “event horizon” in the theory of the black holes in General Relativity (GR) [3]. Note that R_0 happens to be one-half of the Schwartzschild radius R_S . Anyhow the conceptual difference with the event horizon should be kept in mind because in GR the very structure of the space-time is drastically changed beyond R_S and even the “one-way passage” (i.e. the fact that things are free to go inside R_S but never to go outside) is unobtainable in Newtonian physics [4].

However, if we take into account the mass-energy equivalence, we should also take into account the self-energy of the sphere. For example, in classical Newtonian physics, the gravitational energy of a simple spherically symmetric shell of radius R and mass M_0 turns out to be

$$U(R) = -G \frac{M_0^2}{2R} \quad (6)$$

Such a binding energy (negative) is equivalent to a mass defect. Hence the mass of the shell will be different from M_0 . In the following we shall refer to M_0 as the “bare” mass and write $M(R)$ for the “renormalized” mass, i.e. the resulting mass when M_0 is distributed in a spherical shell of radius R . $M(R)$ takes into account the gravitational self-energy, while M_0 corresponds to the sum of all the masses that one would obtain tearing the sphere in many small pieces and moving them away apart. Accordingly, the gravitational potential (1) at the surface ($r = R$) should be written as ²

$$\Phi(R) = -G \frac{M(R)}{R} \quad (7)$$

How to calculate the renormalized mass $M(R)$ from a given bare mass M_0 will be the main point to be discussed in the following.

The necessity of taking into account the self-energy, when treating the problem of a black hole, has been pointed out recently by Christillin [7]. However his correction is valid only at the first order in c^{-2} , or, more precisely, at the first

²The equality of inertial and gravitational masses has been tested experimentally even in presence of mass defects due to large binding energies [5, 6]

order in R_0/R , and cannot be used when $U(R)/c^2$ is comparable to the bare mass M_0 . Here we prove that, taking consistently into account the implications of the mass-energy equivalence and rewriting (2) in terms of the renormalized masses, it is impossible to verify (4) for any finite $R \neq 0$. We could say that, while the implementation of special relativity into Newtonian gravitation allows for a “modern” definition of a NBH, on the other side it denies the possibility of its existence.

2 The consistent mass of a spherical shell and a puzzle

Given the expression (6) for the gravitational energy of a spherical shell, it seems quite natural to write down the following consistent equation for the renormalized mass $M(R)$

$$M(R) = M_0 - \frac{G}{2} \frac{M(R)^2}{Rc^2} \quad (8)$$

whose (positive) solution is

$$M(R) = M_0(-1 + \sqrt{1 + 2R_0/R})R/R_0 \quad (9)$$

This equation has been considered since 1960 [8, 9] in the framework of the classical theory of the electron. In fact, adding to (8) the contribution to the mass of the electromagnetic energy $e^2/2R$ (this time positive), the ensuing solution tends to a finite value when $R \rightarrow 0$: $M(R \rightarrow 0) = |e|/\sqrt{G}$, independent of M_0 . This elegant result exhibits a nice feature of the gravitational self-energy as a regularizing device (unfortunately numerically is too big ($10^{21}m_e$) compared to the electron mass). Instead our interest here is to consider (8) in connection with NBH. From (9) one sees that $M(R)$ goes to zero for $R \rightarrow 0$ as $R^{1/2}$ and that the gravitational potential on the surface of the shell

$$\Phi(R) = -G \frac{M(R)}{R} = -G \frac{M_0}{R_0}(-1 + \sqrt{1 + 2R_0/R}) = -c^2(-1 + \sqrt{1 + 2R_0/R}) \quad (10)$$

goes to $-\infty$ for $R \rightarrow 0$. When $R = 2R_0/3$ one gets

$$\Phi(R = 2R_0/3) = -c^2 \quad (11)$$

Then it seems that taking into account the mass renormalization of the shell, resulting from its self-energy, does not prevent the possibility of existence of a NBH; it will only diminish a bit the value of the radius at which (4) is verified (from R_0 to $2R_0/3$).

However there is a contradiction. Suppose we want to deposit a test particle δm_0 on the surface of $M(R)$ and let us think about this test mass as being uniformly distributed on a thin spherical shell of radius r centered on the origin, just as $M(R)$.

(Note that, neglecting higher orders in δm_0 , we do not worry about self-energy of δm_0 on its own. In other words: $\delta m(r) \approx \delta m_0$.) Now imagine to bring r to R and to stick δm_0 as a thin film on $M(R)$. According to (3), if $R = 2R_0/3$ the total mass of the system should not increase (or even diminish if $R < 2R_0/3$), while according to (9), viewing the system as a new shell of bare mass $M_0 + \delta m_0$, one has

$$M_t(R) = M(R) + \frac{\partial M(R)}{\partial M_0} \delta m_0 = M(R) + \frac{\delta m_0}{\sqrt{1 + 2R_0/R}} > M(R) \quad (12)$$

in clear contradiction. So there is a mistake somewhere.

We conclude this section with an aside remark. Analogous considerations hold for an arbitrary spheric symmetrical distribution of matter. For instance, in the case of a sphere with uniform volume density, one gets a formally identical solution to (9) with the replacement $R_0 \rightarrow R'_0 = 6R_0/5$. The double-shell model that we are exploiting here is most useful since it allows to deal with (radial) pointlike particles.

3 Three recipes for mass renormalization

In order to discover the origin of the discrepancy we should turn back our attention on how to take into account the mutual gravitational interaction energy U_{int} between two bodies of masses M_1, M_2 . Obviously the total mass is

$$M_{tot} = M_1 + M_2 + U_{int}/c^2 \quad (13)$$

but how should we split U_{int} between the two bodies? This point is relevant because U_{int} , in its turn, has to be *consistently* expressed in terms of the modified (fully renormalized) masses. To be specific, let us think of M_1, M_2 as two pointlike bodies at distance r apart and suppose that a fraction x of U_{int}/c^2 be attributed to M_1 (hence a fraction $1 - x$ to M_2), then the autoconsistent expression for U_{int} will be:

$$U_{int} = -G \frac{(M_1 + xU_{int}/c^2)(M_2 + (1 - x)U_{int}/c^2)}{r} \quad (14)$$

which is, in fact, an equation for U_{int} depending on x . In [7] it was suggested to attribute the whole interaction energy to the smaller mass. Actually in the model at hand we considered two concentric shells, the first one with a big mass $M(R)$ (renormalized on its own), the second with an infinitesimal mass δm_0 that works as a test particle. We thought to stick δm_0 on the surface of the first one, keeping spherical symmetry. In this situation, three possible schemes of renormalization are conceivable. In fact the interaction energy between the two shells $\delta U(R)$ may be attributed entirely to the big mass or to the small one, or rather be split in two equal parts between them. In each of these schemes $\delta U(R)$ will assume a specific expression as follows:

1. Renormalization of the big mass $M(R)$:

At 1^o order in δm_0 this further renormalization of $M(R)$ can be neglected in $\delta U(R)$. So

$$\delta U(R) = -G \frac{M(R)\delta m_0}{R} \quad (15)$$

2. Renormalization of the small mass δm_0 :

In this case (2) has to be consistently modified, as specified in (14)

$$\delta U(R) = -G \frac{M(R)(\delta m_0 + \delta U(R)/c^2)}{R} \quad (16)$$

3. Renormalization of both masses by the same amount:

Again at 1^o order in δm_0

$$\delta U(R) = -G \frac{M(R)(\delta m_0 + \delta U(R)/2c^2)}{R} \quad (17)$$

The main point comes along now observing that in the equation for the total mass of the system

$$M_t(R) = M(R) + \delta m_0 + \delta U(R)/c^2 \quad (18)$$

it is

$$M_t(R) - M(R) \equiv dM(R) \quad ; \quad \delta m_0 \equiv dM_0$$

so that (18) is in fact the differential equation that yields the mass $M(R)$ of a spherical shell of radius R as a function of its bare mass M_0 . Each scheme of renormalization leads to a different equation. In the following we shall display the results for each of them.

1. *Renormalization of the big mass $M(R)$*

Given (15), from (18)

$$M_t(R) = M(R) + \delta m_0(1 - G \frac{M(R)}{Rc^2}) \quad (19)$$

we get the differential equation

$$\frac{dM(R)}{dM_0} = 1 - G \frac{M(R)}{Rc^2} \quad (20)$$

whose solution is

$$\ln(1 - G \frac{M(R)}{Rc^2}) = -G \frac{M_0}{Rc^2} \quad (21)$$

$$M(R) = \frac{Rc^2}{G}(1 - \exp[-G \frac{M_0}{Rc^2}]) \equiv M_0(1 - \exp[-\frac{R_0}{R}])\frac{R}{R_0} \quad (22)$$

Therefore in this scheme, the mass of a spherical shell of radius R and bare mass M_0 is not given by (9) (solution of (8)) but by (22). The gravitational potential on the surface is

$$\Phi(R) = -G \frac{M(R)}{R} = -c^2(1 - \exp[-\frac{R_0}{R}]) \quad (23)$$

which keeps finite values and goes to $-c^2$ only at the limit $R \rightarrow 0$.

2. Renormalization of the small mass δm_0

From (16)

$$\delta U(R) = -G \frac{M(R)\delta m_0}{R(1 + G \frac{M(R)}{Rc^2})} \quad (24)$$

i.e. the mass δm_0 , once stuck on $M(R)$, is renormalized as

$$\delta m_0 \rightarrow \delta m = \frac{\delta m_0}{1 + G \frac{M(R)}{Rc^2}} \quad (25)$$

Given (24), from (18)

$$M_t(R) = M(R) + \frac{\delta m_0}{1 + G \frac{M(R)}{Rc^2}} \quad (26)$$

we get the differential equation

$$\frac{dM(R)}{dM_0} = \frac{1}{1 + G \frac{M(R)}{Rc^2}} \quad (27)$$

whose solution is

$$M(R) + G \frac{M(R)^2}{2Rc^2} = M_0 \quad (28)$$

$$M(R) = M_0(-1 + \sqrt{1 + 2R_0/R})R/R_0 \quad (29)$$

Here we recover the (8,9) of Arnowitt, Deser and Missner [8]. Now it is clear the reason of the inconsistency found above: Using (9) one should *coherently* use (24), not (15). This last equation, for $R = 2R_0/3$, would wrongly lead to $\delta U(R) = -c^2\delta m_0$, instead, according to (24), it is $\delta U(R = 2R_0/3) = -c^2\delta m_0/2$ (in agreement with (12)).

The renormalization of the test mass δm_0 may be equivalently described in terms of a suitable modification of the gravitational potential (for $r > R$)

$$\Phi(r) \rightarrow \Phi_r(r) = -\frac{G}{(1 + G \frac{M(R)}{rc^2})} \frac{M(R)}{r} \quad (30)$$

At $r = R$

$$\Phi_r(R) = -c^2(1 - \frac{1}{\sqrt{1 + 2R_0/R}}) \quad (31)$$

Therefore $\Phi_r(R) > -c^2$ for any $R \neq 0$ and $\Phi_r(R) \rightarrow -c^2$ for $R \rightarrow 0$.

3. Renormalization of both masses

From (17) we get

$$\delta U(R) = -G \frac{M(R)\delta m_0}{R(1 + G \frac{M(R)}{2Rc^2})} \quad (32)$$

that means

$$\delta m_0 \rightarrow \delta m = \frac{\delta m_0}{1 + G \frac{M(R)}{2Rc^2}} \quad (33)$$

As in the scheme 2, the renormalization of the particle δm_0 may be equivalently described in terms of a suitable modification of the gravitational potential:

$$\Phi(r) \rightarrow \Phi_r(r) = -\frac{G}{(1 + G \frac{M(R)}{2rc^2})} \frac{M(R)}{r} \quad (34)$$

Given (32), from (18)

$$M_t(R) = M(R) + \frac{1 - G \frac{M(R)}{2Rc^2}}{1 + G \frac{M(R)}{2Rc^2}} \delta m_0 \quad (35)$$

we get the differential equation

$$\frac{dM(R)}{dM_0} = \frac{1 - G \frac{M(R)}{2Rc^2}}{1 + G \frac{M(R)}{2Rc^2}} \quad (36)$$

whose solution will be given by

$$-M(R) - \frac{4Rc^2}{G} \ln(1 - \frac{GM(R)}{2Rc^2}) = M_0 \quad (37)$$

Let $z \equiv \frac{GM(R)}{2Rc^2}$, then (37) may be conveniently put as

$$z = 1 - \exp[-\frac{z}{2}] \exp[-R_0/4R] \approx 1 - \left(1 - \frac{z}{2} + \frac{z^2}{4 \cdot 2!} - \dots\right) \exp[-R_0/4R] \quad (38)$$

It is clear from (37,38) that $0 \leq z < 1$ (for $R \neq 0$), hence (see (35)) $M_t(R) > M(R)$. The modified gravitational potential (34) at $r = R$ turns out to be

$$\Phi_r(R) = -\frac{2zc^2}{1+z} \quad (39)$$

Once again, since $z \rightarrow 1$ for $R \rightarrow 0$, $\Phi_r(R) \rightarrow -c^2$ at that limit.

4 Concluding remarks

In this paper it was shown that, in a Newtonian theory of gravitation that incorporates the mass-energy equivalence, for the interaction energy $\delta U(R)$ between a massive spherically symmetric shell of radius R and a test-particle of mass δm_0 , settled on its surface, it is always (for $R \neq 0$)

$$\delta U(R) > -c^2 \delta m_0 \quad (40)$$

Here $c^2 \delta m_0$ is to be understood as the full relativistic energy of the particle while the subscript on the mass indicates that it is “bare”, i.e. not yet renormalized by the gravitational interaction with the heavy shell. The equation $\delta U(R) \leq -c^2 \delta m_0$ was identified as the condition for the existence of a Newtonian black hole. Then (40) states that a black hole cannot exist in Newtonian gravity.

Besides having taken into account Einstein’s mass-energy equivalence (and the inertial-gravitational mass equality), we wrote

$$\delta U(R) = -G \frac{M(R) \delta m}{R} \quad (41)$$

where $M(R)$ is the mass of the shell that takes into account its own self-energy (the further renormalization of $M(R)$ due to $\delta U(R)$ can be neglected) and δm is the mass of the test particle eventually renormalized by $\delta U(R)$ (the self-energy of δm_0 on its own can be neglected). In fact (40) has been established using three possible recipes for mass renormalization that differ according to the fraction of the interaction energy $\delta U(R)$ that intervenes in the renormalization of δm_0 .

The expression of the renormalized mass $M(R)$ of the shell in terms of its bare mass M_0 is a main achievement of the present paper. Depending upon the scheme of renormalization used, we got three different solutions that yield rather different results at $R \approx R_0$. However they display the same behaviour of $M(R)$ for $R \gg R_0$ (i.e. at the first order in c^{-2}):

$$M(R) \approx M_0(1 - R_0/2R) \quad (42)$$

and, most remarkably, the same lower bound $-c^2 \delta m_0$ as regards $\delta U(R)$ with $\delta U(R) \rightarrow -c^2 \delta m_0$ just at the limit $R \rightarrow 0$. This last result is the main one, since it implies that in no way a NBH could exist.

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